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The Influence of Edges and Corners on Potential Functions of Surface Layers

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Abstract

The potential functions of multiple surface layers behave singularly when approaching the boundary of the surface. The degree of the singularities of these functions and their derivatives are discussed in this paper.

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Let S be an analytic, orientable surface in the three dimensional space; let p and q be two points, $r(p,q) = |p-q|$ the distance between p and q, and $\rho(q)$ a function defined for all $q \in S$. With $\frac{\partial}{\partial n_q}$ we denote the derivative in the normal direction relative to $q \in S$. The potential function of the N-fold layer $\rho(q)$ is then given by

$$U_N(p) = \int_S \rho(q) \left(\frac{\partial}{\partial n_q}\right)^N \frac{1}{r(p,q)} \, dS_q .$$

The functions $U_N(p)$ are analytic in the neighborhood of all points p, not lying on S. $U_N(p)$ jumps, passing with p, through an interior point of S. Papers by Liapounoff, Poincaré and E. Schmidt (cf. [1]) deal with these jump relations. They are explicitly stated for potential functions and their derivatives of arbitrary order in a paper by C. Müller[2]. However, these papers deal only with the behavior of potential functions approaching, with p, an interior point of the surface. But for many problems it is of interest to know the behavior of potential functions and their derivatives when approaching, with p, the boundary C of the surface S. This behavior will be investigated here. We shall see that the potential functions behave singularly when approaching, with p, the boundary C. The degree of the singularity will be given explicitly.

In order not to initially burden the representation with differentiability suppositions, let us for the present assume the surface S and the layer function S to be analytic and the boundary curve C to be piecewise analytic. This assumption will be restricted to differentiability suppositions of finite order in the last section.

The differential geometric formalism which we need will be given in the first section where we follow the notations given in C. Müller's paper [2]. The second section deals with the functions U_o, U_1 and their gradients. In the third section, the singular behavior of the functions U_N and ∇U_N will be reduced to curve integrals over the boundary C. These curve integrals will be discussed in Section 4. Section 5 finally restricts the differentialiability suppositions.

1. Introduction

Let the orientable surface S be bounded by the piecewise analytic Jordan curve C. Let S have the follow ing properties:

1. There exists a coordinate system for every point P of S (a closed domain).
2. P is the origin of the coordinate system.
3. A constant $c > 0$ exists such that the part of S lying in the interior of the sphere $(x^1)^2 + (x^2)^2 + (x^3)^2 \leqq c$ can be represented in the form $x^3 = \phi(x^1, x^2)$. $\phi(x^1,x^2)$ is an analytic function for all $(x^1)^2 + (x^2)^2 < c$.
4. The derivatives $\dfrac{\partial\phi}{\partial x^1} = \phi_{|1}$, $\phi_{|2}$ and $\phi_{|1|2}$ vanish in the origin.

Thus the x^3-axis is normal to S in P; the x^1- and x^2- axis are directions of principal curvature. Therefore, we briefly call this coordinate system a tangent-normal system.

Let e_i be the unit vectors in the direction of the coordinate axis;

$$x^1 e_1 + x^2 e_2 + \phi(x^1, x^2) e_3 = f(x^1, x^2) \tag{1.1}$$

then represents a point of the surface. f denotes a vector with the components f^1, f^2, f^3.

(1.2) $$f_{|\mu} \overset{\circ}{=} e_\mu \ ; \quad f_{|1|2} = \phi_{|1|2}\, e_3 \overset{\circ}{=} 0.$$

In (1.2) and in the following equations, $\overset{\circ}{=}$ denotes equal in the origin (in P); Greek subscripts stand for 1,2; Roman for 1,2,3.

Now we introduce a new coordinate system - in short a u-system - by

(1.3) $$x = f(u^1,u^2) + u^3 n(u^1,u^2)$$

where n denotes the unit vector orthogonal to the surface. Then we have

Then we have

(1.4) $$\frac{\partial x^i}{\partial u^\kappa} \overset{\circ}{=} \delta^i_\kappa$$

and with

(1.5) $$n_{|\mu} = - L^\sigma_\mu\, f_{|\sigma} \ ; \quad \gamma_{\mu\nu} = (f_{|\mu}\, f_{|\nu}) \ ,$$

(1.6) $$\left\{ \begin{aligned} g_{\mu\nu} &= (x_{|\mu}\, x_{|\nu}) = (f_{|\mu} - u^3 L^\sigma_\mu\, f_{|\sigma})(f_{|\nu} - u^3 L^\sigma_\nu\, f_{|\sigma}) \\ &= \gamma_{\mu\nu} - 2u^3 L_{\mu\nu} + (u^3)^2 L^\sigma_\mu L_{\sigma\nu} \ . \end{aligned} \right.$$

Let $2H = k_1 + k_2$ be the mean curvature and $K = k_1 k_2$, the Gauss curvature. Then, with

(1.7) $$L_{11} \overset{\circ}{=} k_1 \ ; \ L_{22} \overset{\circ}{=} k_2 \ ; \ L_{12} \overset{\circ}{=} 0 \ ; \ \gamma_{\mu\nu} = \delta_{\mu\nu} \ ,$$

we have

(1.8) $$g_{\mu\nu} \overset{\circ}{=} (1 - (u^3)^2 K)\, \gamma_{\mu\nu} - 2u^3(1 - u^3 H)\, L_{\mu\nu} \ .$$

Equation (1.8) represents a relation between tensors which is valid in the point P. However, the point P is in no way distinguished on the surface; rather, to every arbitrary point of the surface, there exists a tangent-normal system and, consequently, a u-system in which equation (1.8) is valid. Thus, (1.8) is valid in every point and we get

$$(1.9) \qquad g_{\mu\nu} = (1 - (u^3)^2 K)\, \gamma_{\mu\nu} - 2u^3(1 - u^3 H)\, L_{\mu\nu}$$

and

$$(1.10) \qquad g_{33} = (x_{|3} x_{|3}) = n^2 = 1 \; ; \; g_{3\mu} = (x_{|3} x_{|\mu}) = (n(f_{|\nu} - u^3 L^{\sigma}_{\mu} f_{|\sigma})) = 0.$$

Let g be det g_{ik} and γ be det $\gamma_{\mu\nu}$, then g/γ is an invariant of the surface (u^3 fixed) and we get

$$(1.11) \qquad g \;\hat{=}\; \gamma(1 - 2u^3 H + (u^3)^2 K)^2$$

or with

$$(1.12) \qquad G = 1 - 2u^3 H + (u^3)^2 K$$

and

$$(1.13) \qquad g = \gamma\, G^2 .$$

We get the Delta operator

$$(1.14) \qquad \Delta = \left(\frac{\partial}{\partial x^1}\right)^2 + \left(\frac{\partial}{\partial x^2}\right)^2 + \left(\frac{\partial}{\partial x^3}\right)^2$$

in the u-system by

$$(1.15) \qquad \Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} \sqrt{g}\, g^{ik} \frac{\partial}{\partial u^k}$$

or

(1.16) $$\Delta = \frac{1}{G}\ \frac{1}{\sqrt{\gamma}}\ \frac{\partial}{\partial u^{\mu}}\ G\sqrt{\gamma}\ \ g^{\mu\nu} + \frac{1}{G}\ \frac{\partial}{\partial u^{3}}\ G\ \frac{\partial}{\partial u^{3}}\ .$$

For sufficiently small u^3 we expand

(1.17) $$G\ g^{\mu\nu} = \sum_{j=0}^{\infty} \frac{(u^3)^j}{j\ !}\ S^{\mu\nu}_{(j)}$$

and denote

(1.18) $$\Delta_j = \frac{1}{\sqrt{\gamma}}\ \frac{\partial}{\partial u^{\mu}}\ \sqrt{\gamma}\ \ S^{\mu\nu}_{(j)}\ \frac{\partial}{\partial u^{\nu}} \qquad \left(\Delta_o = \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial u^{\mu}}\ \sqrt{\gamma}\ \ \gamma^{\mu\nu}\frac{\partial}{\partial u^{\nu}}\right).$$

Thus we get

<u>Lemma 1</u>: For sufficiently small u^3 we have

$$G\ \Delta = \sum_{j=0}^{\infty} \frac{(u^3)^j}{j\ !}\ \ \Delta_j + \frac{\partial}{\partial u^3}\ G\ \frac{\partial}{\partial u^3}\ .$$

From Lemma 1, it follows for $u^3 = 0$ and for harmonic functions with

(1.19) $$\frac{\partial}{\partial u^3} = \frac{\partial}{\partial n} \quad \text{on } S$$

(1.20) $$\Delta_o U + \left(\frac{\partial^2}{\partial n^2} - 2H\frac{\partial}{\partial n}\right) U = 0\ .$$

In analogy to Lemma 1, we get, by differention with respect to u^3:

<u>Lemma 2</u>: Let the function U be harmonic in the neighborhood of a point P. Then the derivatives of U satisfy the recurrence relations

$$0 = \left\{ \sum_{j=o}^{N} \binom{N}{j} \Delta_{N-j} \left(\frac{\partial}{\partial n}\right)^{j} + \left(\frac{\partial}{\partial n}\right)^{N+2} - 2H(N+1)\left(\frac{\partial}{\partial n}\right)^{N+1} + 2K\binom{N+1}{2}\left(\frac{\partial}{\partial n}\right)^{N} \right\} U .$$

We get the Nabla operator

$$\nabla = \frac{\partial}{\partial x^1} e_1 + \frac{\partial}{\partial x^2} e_2 + \frac{\partial}{\partial x^3} e_3 \tag{1.21}$$

in the u-system by

$$\nabla U = g^{ki} x_{|i} U_{|k} = g^{\mu\nu} (f_{|\nu} + u^3 n_{|\nu}) U_{|\mu} + nU_{|3} . \tag{1.22}$$

We develop for small values of u^3

$$g^{\mu\nu} (f_{|\nu} + u^3 n_{|\nu}) = g^{\mu\nu} (\delta^{\rho}_{\nu} - u^3 L^{\rho}_{\nu}) f_{|\rho} \tag{1.23}$$

$$= f_{|\rho} \sum_{j=0}^{\infty} \frac{(u^3)^j}{j\,!} Q^{\mu\rho}_{(j)}$$

and use the notation

$$\nabla_j = Q^{\mu\nu}_{(j)} f_{|\nu} \frac{\partial}{\partial u^\mu} \qquad \left(\nabla_o = \gamma^{\mu\nu} f_{|\nu} \frac{\partial}{\partial u^\mu}\right) . \tag{1.24}$$

Then it follows:

<u>Lemma 3</u>: For sufficiently small u^3, we have

$$\nabla = \sum_{j=0}^{\infty} \frac{(u^3)^j}{j\,!} \nabla_j + n \frac{\partial}{\partial u^3}$$

Finally, we want to derive some formulas from Gauss' theorem which we shall need later on. Let J be a continuously differentiable function defined on S. Then we get for a sufficiently small surface S' with the boundary C'

$$(1.25) \qquad \int_{S'} \nabla_j J \, dS = \int_{S'} Q_{(j)}^{\mu\nu} f_{|\nu} J_{|\nu} \sqrt{\gamma} \, du^1 \, du^2$$

$$= - \int_{S'} J \frac{\partial}{\partial u^\mu} \left(Q_{(j)}^{\mu\nu} f_{|\nu} \sqrt{\gamma} \right) du^1 \, du^2$$

$$+ \int_{C'} \sqrt{\gamma} \, J \, f_{|\nu} \left(Q_{(j)}^{1\nu} \dot{u}^2 - Q_{(j)}^{2\nu} \dot{u}^1 \right) dc \, .$$

Since

$$(1.26) \qquad n_j = f_{|\nu} \left(Q_{(j)}^{1\nu} \dot{u}^2 - Q_{(j)}^{2\nu} \dot{u}^1 \right) \sqrt{\gamma} \quad ; \quad p_j = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial u^\mu} \left(Q_{(j)}^{\mu\nu} f_{|\nu} \sqrt{\gamma} \right) ,$$

we find, after summation over S',

<u>Lemma 4:</u> Let J be a continuously differentiable function defined on S. Then we have

$$\int_S \nabla_j J \, dS = - \int_S J \, p_j \, dS + \int_C J \, n_j \, dc \, .$$

For $j = 0$ we have from Lemma 4.

Lemma 4*: Let J be a continuously differentiable function defined in S. Then we have

$$\int_S \nabla_o J \, dS = -2 \int_S n\, HJ \, dS + \int_C n_o J \, dc$$

where n_o is a vector normal to C and orthogonal to n.

Let $\mathcal{H}$ and $\mathcal{L}$ be two times continuously differentiable functions defined on S. Then we have

$$(1.27) \qquad \int_{S'} \mathcal{H} \triangle_j \mathcal{L} \, dS = \int_{S'} \mathcal{H} \frac{\partial}{\partial u^\mu} (\sqrt{\gamma}\, S_{(j)}^{\mu\nu} \mathcal{L}_{|\nu}) du^1 du^2$$

$$= -\int_{S'} \mathcal{H}_{|\mu} \sqrt{\gamma}\, S_{(j)}^{\mu\nu} \mathcal{L}_{|\nu}\, du^1 du^2 + \int_{C'} \sqrt{\gamma}\, \mathcal{H} \mathcal{L}_{|\nu} (S_{(j)}^{1\nu}\, \dot{u}^2 - S_{(j)}^{2\nu}\, \dot{u}^1)\, dc$$

$$= \int_{S'} \mathcal{L} \triangle_j \mathcal{H} \, dS + \int_{C'} \mathcal{H} \sqrt{\gamma}\, \mathcal{L}_{|\nu} (S_{(j)}^{1\nu}\, \dot{u}^2 - S_{(j)}^{2\nu}\, \dot{u}^1)\, dc$$

$$- \int_{C'} \mathcal{L} \sqrt{\gamma}\, \mathcal{H}_{|\mu} (S_{(j)}^{\mu 1}\, \dot{u}^2 - S_{(j)}^{\mu 2}\, \dot{u}^1)\, dc \; .$$

Since

$$(1.28) \qquad s_{(j)}^{\nu} = \sqrt{\gamma}(S_{(j)}^{1\nu}\, \dot{u}^2 - S_{(j)}^{2\nu}\, \dot{u}^1) \; ; \; s_j = s_{(j)}^{\nu} f_{|\nu} \; ; \; \nabla_o \mathcal{L} = \mathcal{L}_{|\nu}\, f_{|\mu}\, \gamma^{\mu\nu}$$

we find

Lemma 5: Let the functions $\mathcal{N}$ and $\mathcal{L}$ be two times continuously differentiable on S. Then we have

$$\int_S \mathcal{N} \triangle_j \mathcal{L}\, dS = \int_S \mathcal{L} \triangle_j \mathcal{N}\, dS + \int_C (\mathcal{N} \nabla_o \mathcal{L} - \mathcal{L} \nabla_o \mathcal{N}) s_j\, dc$$

or, in particular,

Lemma 5*: Let the functions $\mathcal{N}$ and $\mathcal{L}$ be two times continuously differentiable on S. Then we have

$$\int_S \mathcal{N} \triangle_o \mathcal{L}\, dS = \int_S \mathcal{L} \triangle_o \mathcal{N}\, dS + \int_C (\mathcal{N} n_o \nabla_o \mathcal{L} - \mathcal{L} n_o \nabla_o \mathcal{N})\, dc$$

2. The behavior of potential functions arising from single and double layers

The potential function of the single layer $\rho(q)$

(2.1) $$U_o(p) = \int_S \rho(q) \frac{1}{r(p,q)}\, dS_q$$

is continuous everywhere[1]. The behavior of the potential function of the double layer $\mu(q)$

(2.2) $$U_1(p) = \int_S \mu(q) \frac{\partial}{\partial n_q} \frac{1}{r(p,q)}\, dS_q$$

approaching, with p, an interior point of the surface S, is characterized

by the jump relations[1], [2]

$$(2.3)\quad \begin{cases} U_1\big|_{ext.} = 2\pi\mu + \int_S \mu \frac{\partial}{\partial n}\frac{1}{r}\, dS \Big|_{p \in S} \\ U_1\big|_{int.} = -2\pi\mu + \int_S \mu \frac{\partial}{\partial n}\frac{1}{r}\, dS \Big|_{p \in S} \end{cases} .$$

The integral

$$(2.4)\qquad \int_S \mu(q) \frac{\partial}{\partial n} \frac{1}{r(p,q)}\, dS_q \Big|_{p \in S}$$

represents a function continuous for all $p \in S$. Thus $U_1(p)$ is not continuous, passing with p through S, but it is bounded for all p.

The gradients of these functions become singular, however, when approaching, with p, a point of the boundary C. To show this, we form, according to Lemma 3,

$$(2.5)\qquad \nabla U_o = -\int_S \rho(q)\, \nabla_q \frac{1}{r(p,q)}\, dS_q = -\int_S \rho n \frac{\partial}{\partial n}\frac{1}{r}\, dS - \int_S \rho\, \nabla_o \frac{1}{r}\, dS.$$

From this, according to Lemma 4*, we get

$$(2.6)\quad \nabla U_o = -\int_S \rho n \frac{\partial}{\partial n}\frac{1}{r}\, dS + \int_S (\nabla_o \rho)\frac{1}{r}\, dS + 2\int_S nH\rho \frac{1}{r}\, dS - \int_C n_o \rho \frac{1}{r}\, dc .$$

The curve integral in (2.6) becomes logarithmically singular[3] whereas the other surface integrals behave regularly. Thus we get

<u>Theorem 1</u>: The potential function $U_o(p)$ behaves regularly approaching, with p, the boundary C, whereas $\nabla U_o(p)$ becomes logarithmically singular.

We shall now discuss $\nabla U_1(p)$ and, in a similar way, get

$$(2.7) \qquad -\nabla U_1 = \int_S \mu \frac{\partial}{\partial n} \nabla_q \frac{1}{r(p,q)} \, dS_q = \int_S \mu n \frac{\partial^2}{\partial n^2} \frac{1}{r} \, dS$$

$$+ \int_S \mu \nabla_o \frac{\partial}{\partial n} \frac{1}{r} \, dS + \int_S \mu \nabla_1 \frac{1}{r} \, dS \quad .$$

We discuss the resulting surface integrals individually. According to (1.20) and Lemma 5*, we get

$$(2.8) \qquad \int_S \mu n \frac{\partial^2}{\partial n^2} \frac{1}{r} \, dS = 2 \int_S \mu H n \frac{\partial}{\partial n} \frac{1}{r} \, dS - \int_S \mu n \, \Delta_o \frac{1}{r} \, dS$$

$$= 2 \int_S \mu H n \frac{\partial}{\partial n} \frac{1}{r} \, dS - \int_S (\Delta_o \, \mu \, n) \frac{1}{r} \, dS$$

$$- \int_C \left\{ \mu \, n \, (n_o \nabla_o) \frac{1}{r} - \frac{1}{r} (n_o \nabla_o) \, \mu \, n \right\} \, dc \quad .$$

The curve integral behaves singularly, like $\frac{1}{r}$, as p approaches C[3].

According to Lemma 4*, we get, for the second integral in (2.7),

$$(2.9) \qquad \int_S \mu \nabla_o \frac{\partial}{\partial n} \frac{1}{r} \, dS = - \int_S (\nabla_o \mu) \frac{\partial}{\partial n} \frac{1}{r} \, dS - 2 \int_S n \, H \mu \, \frac{\partial}{\partial n} \frac{1}{r} \, dS$$

$$+ \int_C n_o \, \mu \frac{\partial}{\partial n} \frac{1}{r} \, dc \quad .$$

The curve integral in (2.9) also behaves singularly, like $\frac{1}{r}$. Finally, for the third integral in (2.7), according to Lemma 4, we have

$$(2.10)\qquad \int_S \mu\, \nabla_1 \frac{1}{r}\, dS = -\int_S (\nabla_1\mu)\, \frac{1}{r}\, dS - \int_S \mu\, \frac{1}{r}\, p_1\, dS + \int_C \mu\, \frac{1}{r}\, n_o\, dc \quad .$$

Again the curve integral becomes logarithmically singular. Thus we get

Theorem 2: The potential function U_1 is bounded, approaching, with p, the boundary C, whereas ∇U_1 becomes singular like $\frac{1}{r}$ (r being the shortest distance between p and C).

3. The behavior of potential functions of multiple layers

In the last section, we discussed the behavior of potential functions of single and double layers and their derivatives. Now we want to extend these discussions to the potential functions of N-fold layers

$$(3.1)\qquad U_N(p) = \int_S \rho(q) \left(\frac{\partial}{\partial n_q}\right)^N \frac{1}{r(p,q)}\, dS_q \quad .$$

To do this, we first of all want to express the operator $\left(\frac{\partial}{\partial n}\right)^N$ through lower derivatives. We make the assumption

$$(3.2)\qquad \left(\frac{\partial}{\partial n}\right)^N = \overset{1}{\Omega}_N + \overset{2}{\Omega}_N \frac{\partial}{\partial n} \quad ,$$

and, according to Lemma 2, get the recursions

$$(3.3)\qquad \overset{\mu}{\Omega}_{N+2} = 2H\,(N+1)\, \overset{\mu}{\Omega}_{N+1} - 2K \binom{N+1}{2} \overset{\mu}{\Omega}_N - \sum_{j=0}^{N} \binom{N}{j} \Delta_{N-j}\, \overset{\mu}{\Omega}_j$$

with the initial values

$$\overset{1}{\Omega}_o = 1 \; ; \; \overset{2}{\Omega}_o = 0 \; ; \; \overset{1}{\Omega}_1 = 0 \; ; \; \overset{2}{\Omega}_1 = 1 \; . \tag{3.4}$$

It follows from (3.3) and (3.4) that the operators $\overset{1}{\Omega}_{2N}$ and $\overset{1}{\Omega}_{2N+1}$ contain derivatives of maximal order 2N and that the operators $\overset{2}{\Omega}_{2N-1}$ and $\overset{2}{\Omega}_{2N}$ contain derivatives of maximal order 2N-2.

Now we form the adjoint operators $\overset{\mu}{\Phi}_N$ to $\overset{\mu}{\Omega}_N$ according to

$$\int_S \mathcal{N} \overset{\mu}{\Omega}_N \mathcal{L} \, dS = \int_S \mathcal{L} \overset{\mu}{\Phi}_N \mathcal{N} \, dS + \int_C \overset{\mu}{\chi}_N (\mathcal{N}, \mathcal{L}) \, dc \tag{3.5}$$

where $\mathcal{N}$ and $\mathcal{L}$ denote sufficiently differentiable functions. As the operators Δ_{N-j} are self-adjoint, we get, for $\overset{\mu}{\Phi}_N$, the resursion

$$\overset{\mu}{\Phi}_{N+2} = 2(N+1) \overset{\mu}{\Phi}_{N+1} H - 2\binom{N+1}{2} \overset{\mu}{\Phi}_N K - \sum_{j=0}^{N} \binom{N}{j} \overset{\mu}{\Phi}_j \Delta_{N-j} \tag{3.6}$$

with the initial values

$$\overset{1}{\Phi}_o = 1 \; ; \; \overset{2}{\Phi}_o = 0 \; ; \; \overset{1}{\Phi}_1 = 0 \; ; \; \overset{2}{\Phi}_1 = 1 \; . \tag{3.7}$$

According to Lemma 5, we get, for the $\overset{\mu}{\chi}_N$,

$$\begin{aligned} \overset{\mu}{\chi}_{N+2} = {} & 2(N+1) \overset{\mu}{\chi}_{N+1} (H\mathcal{N}, \mathcal{L}) - 2\binom{N+1}{2} \overset{\mu}{\chi}_N (K\mathcal{N}, \mathcal{L}) \\ & - \sum_{j=0}^{N} \binom{N}{j} \Big\{ (\mathcal{N} \nabla_o \overset{\mu}{\Omega}_j \mathcal{L} - (\overset{\mu}{\Omega}_j \mathcal{L}) \nabla_o \mathcal{N}) \, s_{n-j} \\ & + \overset{\mu}{\chi}_j (\Delta_{N-j} \mathcal{N}, \mathcal{L}) \Big\} \end{aligned} \tag{3.8}$$

with the initial values.

$$(3.9) \qquad \overset{1}{\chi}_o = \overset{1}{\chi}_1 = \overset{2}{\chi}_o = \overset{2}{\chi}_1 = 0 \; .$$

From (3.8) and (3.9) it follows that $\overset{1}{\chi}_{2N}(\mathcal{H},\mathcal{L})$ and $\overset{1}{\chi}_{2N+1}(\mathcal{H},\mathcal{L})$ contain derivatives of maximal order 2N-1 with reference to $\mathcal{L}$, and that $\overset{2}{\chi}_{2N-1}(\mathcal{H},\mathcal{L})$ and $\overset{2}{\chi}_{2N}(\mathcal{H},\mathcal{L})$ contain derivatives of maximal order 2N-3 with reference to $\mathcal{L}$. $\overset{\mu}{\chi}_{2N}(\mathcal{H},\mathcal{L})$ and $\overset{\mu}{\chi}_{2N+1}(\mathcal{H},\mathcal{L})$ contain derivatives of maximal order 2N-1 with with reference to $\mathcal{H}$.

Thus we get

<u>Lemma 6</u>: Let the function $\mathcal{H}$ be analytic and $\mathcal{L}$ be harmonic. Then there exist operators $\overset{\mu}{\Omega}_N$, $\overset{\mu}{\Phi}_N$, $\overset{\mu}{\chi}_N$ defined in (3.3,6,8) with

$$\left(\frac{\partial}{\partial n}\right)^N \mathcal{L} = \left(\overset{1}{\Omega}_N + \overset{2}{\Omega}_N \frac{\partial}{\partial n}\right)\mathcal{L}$$

and

$$\int_S \mathcal{H}\,\overset{\mu}{\Omega}_N\,\mathcal{L}\,dS = \int_S \mathcal{L}\,\overset{\mu}{\Phi}_N\,\mathcal{L}\,dS + \int_C \overset{\mu}{\chi}_N(\mathcal{L},\mathcal{H})\,dc \quad .$$

It follows from Lemma 6 that

$$(3.10) \qquad U_N = \int_S \rho\left(\frac{\partial}{\partial n}\right)^N \frac{1}{r}\,dS = \int_S \rho\,\overset{1}{\Omega}_N \frac{1}{r}\,dS + \int_S \rho\,\overset{2}{\Omega}_N \frac{\partial}{\partial n}\,\frac{1}{r}\,dS$$

$$= \int_S \left(\overset{1}{\Phi}_N\,\rho\right)\frac{1}{r}\,dS + \int_S \left(\overset{2}{\Phi}_N\,\rho\right)\frac{\partial}{\partial n}\,\frac{1}{r}\,dS$$

$$+ \int_C \overset{1}{\chi}_N\left(\rho,\frac{1}{r}\right)dc + \int_C \overset{2}{\chi}_N\left(\rho,\frac{\partial}{\partial n}\,\frac{1}{r}\right)dc \quad .$$

Thus, we have expressed the function U_N through functions of the type U_o and U_1 and boundary integrals. These curve integrals in (3.10) therefore describe the singular behavior of the functions $U_N(p)$ approaching, with p, the boundary C. We shall discuss the singular behavior of these integrals in the next section.

For the gradient ∇U_N we get, according to Lemmas 3 and 4,

$$(3.11)\qquad -\nabla U_N = \int_S \rho\left(\frac{\partial}{\partial n}\right)^N \nabla \frac{1}{r}\, dS = \int_S \rho n\left(\frac{\partial}{\partial n}\right)^{N+1} \frac{1}{r}\, dS$$

$$+ \sum_{j=0}^{N} \int_S \rho\, \nabla_{N-j} \left(\frac{\partial}{\partial n}\right)^j dS$$

$$= \int_S \rho n\left(\frac{\partial}{\partial n}\right)^{N+1} \frac{1}{r}\, dS - \sum_{j=0}^{N} \int_S \left(\nabla_{N-j}\, \rho\right)\left(\frac{\partial}{\partial n}\right)^j \frac{1}{r}\, dS$$

$$- \sum_{j=0}^{N} \left\{ \int_S p_j \rho\left(\frac{\partial}{\partial n}\right)^j \frac{1}{r}\, dS - \int_C n_j \rho\left(\frac{\partial}{\partial n}\right)^j \frac{1}{r}\, dc \right\}.$$

The surface integrals in (3.11) are again functions of the form U_K. We have now expressed their singular behavior in terms of curve integrals which we shall discuss in the next paragraph.

4. The singular behavior of potential functions arising from one-dimensional layers

In the last section we expressed the singular behavior of potential functions of surface layers through curve integrals over the boundary C of the surface S. We shall now discuss these integrals. As the derivates with respect to all coordinates u^i enter into the integrands, we shall discuss in particular

$$(4.1) \qquad V_N = \int_C \rho \left(\frac{\partial}{\partial c}\right)^N \frac{1}{r}\, dc$$

and

$$(4.2) \qquad W_N = \int_C \rho \left(\frac{\partial}{\partial n}\right)^N \frac{1}{r}\, dc$$

where c denotes the arclength and n a vector normal to C.

We assumed C to be piecewise analytic, so let us assume that $C = \sum_{i=1}^{K} C_i$, where the C_i are analytic curves. Through partial integration, it follows from (4.1) that

$$(4.3) \qquad V_N = \sum_i \int_{C_i} \rho \left(\frac{\partial}{\partial c_i}\right)^N \frac{1}{r}\, dc = \sum_i \left\{ \left[\rho\left(\frac{\partial}{\partial c_i}\right)^{N-1} \frac{1}{r}\right]_0^{L_i} - \int_{C_i} \left(\frac{\partial}{\partial c_i}\right) \rho \left(\frac{\partial}{\partial c_i}\right)^{N-1} \frac{1}{r}\, dc_i \right\}$$

or

$$(4.4) \qquad V_N = \sum_i \left\{ -\sum_{j=1}^{N} (-1)^j \left[\left(\frac{\partial}{\partial c_i}\right)^{j-1} \rho \left(\frac{\partial}{\partial c_i}\right)^{N-j} \frac{1}{r}\right]_0^{L_i} + (-1)^N \int_{C_i} \left(\frac{\partial}{\partial c_i}\right)^N \rho\, \frac{1}{r}\, dc_i \right\}.$$

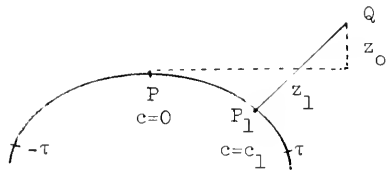
Thus we get [3]

Lemma 7: The function $V_N(p)$ becomes logarithmically singular when approaching C, with p. For $N \geq 1$ there are additional singularities at the corners of C of the order r^{-N} (r being the distance from p to a corner).

To discuss the behavior of the functions W_N, we represent the curve C in the neighborhood of an interior point P in a tangent-normal coordinate system in the following way:

1. P is the origin of the coordinate system.
2. A constant $\tau > 0$ exists so that the part of C, lying in the interior of the sphere $x^2 + y^2 + z^2 \leq \tau$, can be represented in the form $y = f(x)$, $z = g(x)$. The functions f and g are analytic for $x^2 < \tau$.
3. The derivatives f' and g' vanish at the origin.
4. In P the direction of the normal vector n is the direction of the z-axis.

If P is a corner point of C, we can continue the analytic pieces of C over P, so that P again may be considered as an interior point (for each segment separately).



To discuss the function W_1, approaching with $Q(x_o, y_o, z_o)$ the point $P(o,o,o)$, we proceed in the following way: Let Q be sufficiently near to P. Then choose a point $P_1(c = c_1)$ so that Q has the coordinates (o, y_1, z_1) in the tangent-normal system to P_1. Let P_1 lie in a τ_1-neighborhood of $P_o (\tau_1 < \tau)$. Then we can write W_1 in the form

$$(4.5) \qquad W_1 = \overset{\tau_1}{W_1} + \overset{*}{W_1} \quad \text{with} \quad \overset{\tau_1}{W_1} = \int_{-\tau_1}^{\tau_1} \rho \frac{\partial}{\partial n} \frac{1}{r} \, ds.$$

The function $\overset{*}{W_1}$ is bounded; for $\overset{\tau_1}{W_1}$ we get

$$(4.6) \quad |W_1^{\tau_1}| \leq A \left| \int_{-\tau_1}^{\tau_1} \frac{\bar{n}\,\bar{r}}{r^3}\, dx \right| = \mathcal{O}\left(\int_{-\tau_1 - c_1}^{\tau_1 - c_1} \frac{x^2 + x(y - y_1) + (z - z_1)}{\sqrt{x^2 + (y - y_1)^2 + (z - z_1)^2}^{\,3}}\, dx \right)$$

$$= \mathcal{O}\left(\int_{-\tau_1 - c_1}^{\tau_1 - c_1} \frac{dx}{r^2} \right) \quad .$$

With

$$(4.7) \quad r = \sqrt{x^2 + (y - y_1)^2 + (z - z_1)^2} \quad ; \quad \rho = \sqrt{y_1^2 + z_1^2} \quad ; \quad R = \sqrt{x^2 + \rho^2}$$

we get

$$(4.8) \quad \left| \frac{1}{r} - \frac{1}{R} \right| = \frac{|R^2 - r^2|}{rR(r + R)} \quad \frac{|y|(|y - y_1| + |y_1|) + |z|(|z - z_1| + |z_1|)}{rR(r + R)}$$

$$= \mathcal{O}(1)$$

and, by induction, it follows that

$$(4.9) \qquad \left| \frac{1}{r^N} - \frac{1}{R^N} \right| = \mathcal{O}\left(\frac{1}{R^{N-1}} \right) .$$

Thus we get

$$(4.10) \qquad |W_1^{\tau_1}| = \mathcal{O}\left(\int \frac{dx}{R^2} \right) = \mathcal{O}\left(\frac{1}{\rho} \right),$$

ρ being the distance of Q from the curve C (in the normal plane). The example of the straight line shows that in general the estimate in (4.10) cannot be improved.

In a similar way, we get, in the general case ($N \geq 1$)

$$(4.11) \qquad |W_N^{\tau_1}| = \mathcal{O}\left(\int \frac{dx}{R^{N+1}}\right) = \mathcal{O}\left(\frac{1}{\rho^N}\right) .$$

Thus we have

<u>Lemma 8</u>: The function $W_N(\rho)$ becomes singular, like ρ^{-N}, if p approaches the curve C. ρ is the distance from p to the curve C taken in the normal plane of the curve.

We can now describe the singular behavior of the functions

$$(4.12) \qquad U_N = \int_S \rho \left(\frac{\partial}{\partial n}\right)^N \frac{1}{r} \, dS .$$

As the curve integrals in (3.10) contain derivatives of $\frac{1}{r}$ of the maximal order N-1, we get

<u>Theorem 3</u>:. The potential functions

$$U_N(p) = \int_S \rho \left(\frac{\partial}{\partial n}\right)^N \frac{1}{r} \, dS \qquad N > 1$$

become singular, like ρ^{-N}, if p approaches the curve C (ρ being the distance from p to the curve C taken in the normal plane of the curve). Besides this, there are point singularities at the corners of order ρ_1^{1-N} (ρ_1 being the distance from p to the corner).

Similarly, from (3.11), we get

Theorem 4: The potential functions

$$\nabla U_N = \nabla \int_S \rho \left(\frac{\partial}{\partial n}\right)^N \frac{1}{r} \, dS \qquad N \geq 1$$

become singular, like ρ^{-N}, if p approaches the curve C. There are point singularities at the corners of order ρ_1^{-N}.

5. The reduction of the differentiability assumptions

In order to simplify the presentation, we have so far assumed an analytic behavior for the surface S, for the boundary curve C and for the layer function ρ. However, this analytic behavior was not used. Actually, it is sufficient to make differentiability assumptions of finite order. While discussing the behavior of the functions U_N in interior points of S these assumptions are stated in [2]. Thus, let us confine ourselves to boundary points. In discussing the functions U_N, only derivative of order N-1 occurred. Thus, we assume C,S and ρ to be (N-1)-times differentiable. To be able to introduce the tangent-normal coordinate system, however, C has to be at least twice (or one-times Hölder-continuously) differentiable. The layer function ρ has at least to be bounded. For the discussion of ∇U, the derivatives of order N should exist.

. . .

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